

TWISTED CONJUGACY CLASSES IN LATTICES IN SEMISIMPLE LIE GROUPS

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ABSTRACT. Given a group automorphism $\phi : \Gamma \longrightarrow \Gamma$, one has an action of Γ on itself by ϕ -twisted conjugacy, namely, $g.x = gx\phi(g^{-1})$. The orbits of this action are called ϕ -conjugacy classes. One says that Γ has the R_∞ -property if there are infinitely many ϕ -conjugacy classes for every automorphism ϕ of Γ . In this paper we show that any irreducible lattice in a connected non-compact semi simple Lie group having finite centre and rank at least 2 has the R_∞ -property.

1. INTRODUCTION

Let Γ be a finitely generated infinite group and let $\phi : \Gamma \longrightarrow \Gamma$ be an endomorphism. One has an equivalence relation \sim_ϕ on Γ defined as $x \sim_\phi y$ if there exists a $g \in \Gamma$ such that $y = gx\phi(g)^{-1}$. The equivalence classes are called the ϕ -conjugacy classes. Note that when ϕ is the identity, ϕ -conjugacy classes are the usual conjugacy classes. The ϕ -conjugacy classes are nothing but the orbits of the action of Γ on itself defined as $g.x = gx\phi(g^{-1})$. The ϕ -conjugacy class containing $x \in \Gamma$ is denoted $[x]_\phi$ or simply $[x]$ when ϕ is clear from the context. The set of all ϕ -twisted conjugacy classes is denoted by $\mathcal{R}(\phi)$. The cardinality $R(\phi)$ of $\mathcal{R}(\phi)$ is called the *Reidemeister number* of ϕ . One says that Γ has the R_∞ -property for automorphisms (more briefly, R_∞ -property) if there are infinitely many ϕ -twisted conjugacy classes for every automorphism ϕ of Γ . If Γ has the R_∞ -property, we shall call Γ an R_∞ -group.

The notion of twisted conjugacy originated in Nielson-Reidemeister fixed point theory and also arises in other areas of mathematics such as representation theory, number theory and algebraic geometry. See [3] and the references therein. The problem of determining which classes of groups have R_∞ -property is an area of active research initiated by Fel'shtyn and Hill [5].

Let G be a non-compact semi simple Lie group with finite centre. Recall that a discrete subgroup $\Gamma \subset G$ is called a *lattice* if G/Γ has a finite G -invariant measure. One says that Γ is *cocompact* if G/Γ is compact; otherwise Γ is non-cocompact. If, for any non-compact closed normal subgroup $H \subset G$, the image of Γ under the quotient map $G \longrightarrow G/H$ is dense, one says that Γ is irreducible. If G has no compact factors, Γ is irreducible if and only if for any two closed normal subgroups H_1, H_2 of G such that $G = H_1.H_2$ and lattices

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$\Gamma_i \subset H_i$, the group $\Gamma_1 \Gamma_2$ is not commensurable with Γ . In particular, any lattice in G is irreducible if G is simple.

The main result of this paper is the following:

Theorem 1.1. *Let Γ be any irreducible lattice in a connected semi simple non-compact Lie group G with finite centre. If the real rank of G is at least 2, then Γ has the R_∞ property.*

When G has real rank 1, the above result is well-known. Indeed, assume that G has real rank 1. When the lattice Γ is cocompact, it is hyperbolic. When Γ is not cocompact, it is relatively hyperbolic. It has been shown by Levitt and Lustig [7] that any torsion free non-elementary hyperbolic group has the R_∞ -property. Fel'shtyn ([2],[3]) established the R_∞ property for arbitrary non-elementary hyperbolic groups as well as non-elementary relatively hyperbolic groups.

When Γ is a principal congruence subgroup of $\mathrm{SL}(n, \mathbb{Z})$, the above theorem was established in [10]. When $\Gamma = \mathrm{Sp}(2n, \mathbb{Z})$, the result was first proved by Fel'shtyn and Gonçalves [4]; see also [10].

Our proof of the above theorem involves only elementary arguments, using some well-known but deep results concerning irreducible lattices in semi simple Lie groups. The main theorem is first established when G has no compact factors and has trivial centre. In this case, the proof uses the Zariski density property of Γ due to Borel as well as the strong rigidity theorem. When G has non-trivial compact factors, we need to use Margulis' normal subgroup theorem to reduce to the case when G has trivial centre and no compact factors.

In §2 we shall recall the results on lattices in semi simple Lie groups needed in the proof of Theorem 1.1, given in §3.

2. LATTICES IN SEMI SIMPLE LIE GROUPS

We recall below the definition of an arithmetic lattice in a semi simple Lie group and some deep results concerning them relevant for our purposes.

Let $\mathbf{G} \subset \mathrm{GL}(n, \mathbb{C})$ be an algebraic group, that is, \mathbf{G} is a subgroup of $\mathrm{GL}(n, \mathbb{C})$ such that \mathbf{G} is the zero locus of a collection of (finitely many) polynomial equations $f_m(X_{ij}) = 0$ in the n^2 matrix entries $X_{i,j}$, $1 \leq i, j \leq n$. One says that \mathbf{G} is defined over a subfield $k \subset \mathbb{C}$ if the f_m can be chosen to have coefficients in k ; in this case $G_k := \mathbf{G} \cap \mathrm{GL}(n, k)$ is the k -points of \mathbf{G} . If R is a subring of k , then $G_R := \mathbf{G} \cap \mathrm{GL}(n, R)$. A theorem of Borel and Harish-Chandra asserts that if \mathbf{G} is a connected semi simple algebraic group defined over \mathbb{Q} then $G_{\mathbb{Z}}$ is a lattice in $G_{\mathbb{R}}$. We say that a lattice $\Gamma \subset G$ is *arithmetic* if \mathbf{G} is defined over \mathbb{Q} and if Γ is commensurable with $G_{\mathbb{Z}}$.

If $N \subset G$ is a compact normal subgroup of a connected Lie group G with finite centre and Γ a discrete subgroup of G , then Γ is a lattice in G if and only if the image of Γ under the quotient map $G \rightarrow G/N$ is a lattice.

Let G be a connected semi simple Lie group and $K \subset G$ a maximal compact subgroup. The *real rank* of G is the largest integer m such that the Euclidean space \mathbb{R}^m can be imbedded as a totally geodesic submanifold of the symmetric space G/K . Equivalently, the real rank of G is the dimension of the largest abelian subalgebra contained in \mathfrak{p} where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition. Here $\mathfrak{g} = \text{Lie}(G)$, $\mathfrak{k} = \text{Lie}(K)$.

The following well-known results will be needed in the proof of our main theorem.

Theorem 2.1. (Borel density theorem) *Let $\Gamma \subset G_{\mathbb{R}}$ be any lattice in a connected semi simple algebraic group \mathbf{G} defined over \mathbb{Q} . If $G_{\mathbb{R}}$ has no compact factors, then Γ is Zariski dense in \mathbf{G} .* \square

Theorem 2.2. (Margulis' normal subgroup theorem) *Let $\Gamma \subset G$ be an irreducible lattice where G is a connected semi simple Lie group of rank at least 2 and with finite centre. If N is normal in Γ , then either N is of finite index in Γ or is a finite subgroup contained in the centre of G .* \square

Next we state the strong rigidity for irreducible lattices.

Theorem 2.3. (Strong rigidity) *Let G and G' be connected linear semi simple Lie groups with trivial centre and having no compact factors. Let $\Gamma \subset G$ and $\Gamma' \subset G'$ be irreducible lattices. Assume that G and G' are not locally isomorphic to $SL(2, \mathbb{R})$. Then any isomorphism $\phi : \Gamma \rightarrow \Gamma'$ extends to an isomorphism $G \rightarrow G'$ of Lie groups.* \square

The strong rigidity theorem for cocompact lattices was obtained by Mostow [9]. Margulis showed that the result holds for G as above with real rank ≥ 2 . The rank 1 case (when the lattice is non-cocompact) is due to Prasad [12], who extended the classical work of Mostow concerning rigidity of rank 1 compact locally symmetric manifolds. The proofs of the rigidity theorem for the case rank ≥ 2 , the Borel density theorem, and the Margulis' normal subgroup theorem can be found in [13].

3. PROOF OF THEOREM 1.1

Before we begin the proof, we recall some elementary notions in combinatorial group theory and recall some facts concerning the R_{∞} -property.

Let Γ be a group and H a subgroup of Γ . Recall that a subgroup H is said to be *characteristic* in Γ if $\phi(H) = H$ for every automorphism ϕ of Γ . Γ is called *hopfian* (resp. *co-hopfian*) if every surjective (resp. injective) endomorphism of Γ is an automorphism of Γ . One says that Γ is *residually finite* if, given any $g \in \Gamma$, there exists a finite index subgroup H in Γ such that $g \notin H$. It is well-known that any finitely generated subgroup of $GL(n, k)$, where k is any field, is residually finite and that finitely generated residually finite groups are hopfian. We refer the reader to [8] for detailed discussion on these notions.

We recall here some facts concerning the R_{∞} -property. Let

$$1 \longrightarrow N \xrightarrow{j} \Lambda \xrightarrow{\eta} \Gamma \longrightarrow 1 \tag{1}$$

be an exact sequence of groups.

Lemma 3.1. *Suppose that N is characteristic in Λ and that Γ has the R_∞ -property, then Λ also has the R_∞ -property.*

Proof. Let $\phi : \Lambda \rightarrow \Lambda$ be any automorphism. Since N is characteristic, $\phi(N) = N$ and so ϕ induces an automorphism $\bar{\phi} : \Gamma \rightarrow \Gamma$. Since $R(\bar{\phi}) = \infty$, it follows that $R(\phi) = \infty$. \square

The following proposition is perhaps well-known; a proof can be found in [10].

Proposition 3.2. *Let Γ be a countably infinite residually finite group. Then $R(\phi) = \infty$ for any inner automorphism ϕ of Γ .* \square

We are now ready to prove the main theorem.

Proof of Theorem 1.1: First suppose that G has trivial centre and has no compact factors. Since the centre of G is trivial, the homomorphism $\iota : G \rightarrow \text{Aut}(G)$ given by inner automorphism allows us to identify G with the group of inner automorphisms of G . Under this identification, G is the identity component of $\text{Aut}(G)$ and $\text{Aut}(G)/G \cong \text{Out}(G)$ is finite. Also the group $\text{Aut}(G)$ is isomorphic to the linear Lie group $\text{Aut}(\mathfrak{g}) \subset \text{GL}(\mathfrak{g})$ of the automorphisms of the Lie algebra \mathfrak{g} of G under which $\phi \in \text{Aut}(G)$ corresponds to its derivative at the identity element. Thus we have a chain of monomorphisms $\Gamma \hookrightarrow G \xrightarrow{\iota} \text{Aut}(G) \cong \text{Aut}(\mathfrak{g}) \hookrightarrow \text{GL}(\mathfrak{g})$. Furthermore, $\text{Aut}(G) \cong \text{Aut}(\mathfrak{g})$ is the \mathbb{R} -points of the complex algebraic group $\mathbf{H} := \text{Aut}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$ and the identity component of $H_{\mathbb{R}}$ is $\text{Aut}(G)^0 = G$.

Suppose that $\phi : \Gamma \rightarrow \Gamma$ is an automorphism. Clearly $\phi \circ \iota_\gamma = \iota_{\phi(\gamma)} \circ \phi$ where ι_γ denotes conjugation by γ . Now let $x, y \in \Gamma$ be such that $x \sim_\phi y$. Then there exists a $\gamma \in \Gamma$ such that $y = \gamma x \phi(\gamma^{-1})$; equivalently, $\iota_y = \iota_\gamma \iota_x \iota_{\phi(\gamma)^{-1}} = \iota_\gamma \iota_x \phi \iota_{\gamma^{-1}} \phi^{-1}$. Hence $\iota_y \phi = \iota_\gamma (\iota_x \phi) \iota_{\gamma^{-1}}$.

By the strong rigidity theorem, $\phi \in \text{Aut}(\Gamma)$ extends to an automorphism of the Lie group G , again denoted $\phi \in \text{Aut}(G)$. For any $h \in H_{\mathbb{R}}$, consider the function $\tau_h : \mathbf{H} \rightarrow \mathbb{C}$ defined as $\tau_h(x) = \text{tr}(xh)$, the trace of $xh \in \mathbf{H} \subset \text{GL}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$. Clearly this is a morphism of varieties defined over \mathbb{R} . We have that, if $x, y \in \Gamma$, $x \sim_\phi y$, then $\tau_\phi(y) = \tau_\phi(x)$ since $\iota_y \phi$ and $\iota_x \phi$ are conjugates in \mathbf{H} .

Assume that the Reidemeister number of ϕ is finite. Then, by what has been observed above, τ_ϕ assumes only finitely many values on $\Gamma \subset H_{\mathbb{R}}^0 = G$. Since, by the Borel density theorem, Γ is Zariski dense in \mathbf{H}^0 , it follows that τ_ϕ is constant on \mathbf{H}^0 . This clearly implies that $\tau_{h\phi}$ is constant for any $h \in H_{\mathbb{R}}^0$.

Let K be a maximal compact subgroup of $H_{\mathbb{R}} = \text{Aut}(G)$. Since $\text{Aut}(G)$ has only finitely many components, by a well-known result of Mostow, K meets every connected component of $\text{Aut}(G)$. (See [1, Theorem 1.2, Ch. VII], [6].) Thus K contains representatives of every element of $\text{Out}(\Gamma)$ and so we may choose an $h \in H_{\mathbb{R}}^0$ such that $\theta := h\phi \in K$. The automorphism $\text{Ad}(\theta)$ on the Lie algebra $\text{Lie}(K^0)$ fixes a regular (semi simple) element $X \in \text{Lie}(K^0)$ by §3.2, Ch. VII of [1]. Hence the one-parameter subgroup $S := \{\exp(tX) \mid t \in \mathbb{R}\} \subset K^0$ is contained in the centralizer $C_{H_{\mathbb{R}}}(\theta) = \{x \in H_{\mathbb{R}} \mid \theta x = x\theta\}$. Note that

θ is also semi simple since K is compact subgroup of $GL(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$. It follows that θ and $\exp(tX), t \in \mathbb{R}$, are simultaneously diagonalizable (over \mathbb{C}). It is now readily seen that τ_{θ} is not constant on $S \subset H_{\mathbb{R}}^0$, a contradiction to our earlier observation that $\tau_{h\phi}$ is a constant function for any $h \in H_{\mathbb{R}}^0$. This implies that $R(\phi) = \infty$.

Next suppose that G has no compact factors but possibly with non-trivial centre, Z . By our hypothesis Z is finite. Clearly $Z \cap \Gamma \subset Z(\Gamma)$ the centre of Γ . Since $\bar{\Gamma} := \Gamma/(Z \cap \Gamma)$ is Zariski dense in G/Z , and since G/Z has trivial centre, we see that $\Gamma/(Z \cap \Gamma)$ has trivial centre. It follows that $Z(\Gamma) = Z \cap \Gamma$. Consider the exact sequence

$$1 \rightarrow Z \cap \Gamma \rightarrow \Gamma \rightarrow \bar{\Gamma} \rightarrow 1 \quad (2)$$

Since $Z \cap \Gamma = Z(\Gamma)$ is a finite characteristic subgroup of Γ , the R_{∞} property for Γ follows from that for $\bar{\Gamma}$.

Finally let G be any Lie group as in the theorem. Let M be the maximal compact normal subgroup of G . Note that M contains the centre Z of G . Now $M \cap \Gamma$ is a *finite* normal subgroup of Γ . We invoke Theorem 2.2 to conclude that $M \cap \Gamma$ is contained in the centre of G . Also $Z(\Gamma)$ is contained in Z since, otherwise, by Theorem 2.2 again, Γ would be virtually abelian. Since G is a non-compact semi simple Lie group, this is impossible. Since M contains Z , we see that $M \cap \Gamma = Z \cap \Gamma$ equals the centre of Γ and hence is characteristic in Γ . Now $\bar{\Gamma} := \Gamma/(M \cap \Gamma)$ is an irreducible lattice in G/M , which has trivial centre and no compact factors. Using the exact sequence (2) again, we see that $R(\phi) = \infty$. This completes the proof. \square

Remark 3.3. (i) Suppose that G is not locally isomorphic to $SL(2, \mathbb{R})$ and that the real rank of G equals 1. When G has no compact factors, the above proof can be repeated verbatim to show that Γ has the R_{∞} property. When G has compact factors and Γ is residually finite (for example when G is linear) one can find a finite index characteristic subgroup Γ' of Γ such that $\Gamma' \cap M = \{1\}$ where M is as in the above proof. Now $\Gamma' \cong \Gamma'/M$ and so has the R_{∞} property. It follows from Lemma 2.2 of [10] that Γ has the R_{∞} property.

(ii) Suppose that G is a linear connected semi simple Lie group of real rank at least 2 and let Γ be an irreducible lattice in G . Since Γ is finitely generated and linear, it follows that Γ is residually finite and hence Hopfian. Let $1 \rightarrow A \xrightarrow{j} \Lambda \xrightarrow{\eta} \Gamma \rightarrow 1$ be an exact sequence of groups where A is any countable abelian group. Proceeding as in the proof of [10, Theorem 1.1(ii)], one can show that Λ has the R_{∞} -property. We give an outline of the proof. Let $\phi \in \text{Aut}(\Lambda)$ and let $f = \eta \circ \phi|_A$. Then $f(A)$ is normal in Γ . By the normal subgroup theorem of Margulis, either $f(A)$ is of finite index—in which case $f(A)$ is a lattice in G —or $f(A)$ is contained in the centre of G since G has real rank at least 2 and Γ is irreducible. Since Γ is not virtually abelian, we see that $f(A)$ has to be finite. Replacing A by $\tilde{A} := \eta^{-1}(Z(\Gamma))$ we see that \tilde{A} is a characteristic subgroup of Λ . Using the observation that Γ is Hopfian and proceeding as in [10], we see that Λ has the R_{∞} property.

(iii) Timur Nasibullov [11] has obtained the following result. Let $\Gamma = GL(n, R)$ or

$SL(n, R)$, $n \geq 3$, where R is an infinite integral domain and let Φ be the subgroup of $Aut(\Gamma)$ generated by the inner automorphisms, homothety by a central character, and the contragradient automorphisms. Then for any $\phi \in \Phi$, one has $R(\phi) = \infty$. In particular, if R has no non-trivial automorphism (e.g. $R = \mathbb{R}$) and has characteristic zero, then Γ has the R_∞ -property.

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